20^{th} BALKAN MATHEMATICAL OLYMPIAD
2003

1. Is there a set of 4004 positive integers so that the sum of any 2003 of them be not divisible by 2003?

   Proof: This problem is very easy. Let $n = 2003, 2n - 2 = 4004$. Let $n - 1$ numbers be $1 \bmod n$ and $n - 1$ numbers be $2 \bmod n$ (distinct numbers). Out of any n numbers, a are equal to $2 \bmod n$ and $n - a$ are equal to $1 \bmod n$. Their sum is $a \bmod n$ and since $1 \leq a \leq n - 1$ you are done. ■

   Observation: It is a classical problem that there are no such numbers if we replace $2n - 2$ by $2n - 1$. See, e.g., Yaglom.

2. Let ABC be a triangle with $AB \neq AC$, and let D be the point of intersection between the tangent at A to the circumcircle of ABC and BC. Consider the points E, F which lie on the perpendiculars raised from B and C to BC, and on the perpendicular bisectors of AB and AC, respectively. Prove that D,E and F are collinear.

   Proof: This problem is cute. Let $\mathcal{E}, \mathcal{F}$ be the circles of centers E, F and tangent to BC and going through A. This can be done by definition.

   Take the inversion of pole D that invaries A and then swaps B and C (by power of a point). The inversion also invaries DA, BC. $\mathcal{E}$ and $\mathcal{F}$ will be circles, tangent at B and C to BC, so they basically swap positions via inversion. But then we are done, because $\mathcal{E}$ and $\mathcal{F}$ are inverses of one another. (To be uselessly more precise, the other tangent from D to $\mathcal{E}$ (other than BC) must meet $\mathcal{F}$ at one point, the inverse of the tangent point. Then the two common tangents to $\mathcal{E}$ and $\mathcal{F}$ meet at D and we are done.) ■

3. Find all functions $f : \mathbb{Q} \longrightarrow \mathbb{R}$ that satisfy the conditions:

   (a) $f(1) + 1 > 0$.

   (b) $f(x + y) - xf(y) - yf(x) = f(x)f(y) - x - y + xy$, for all $x, y \in \mathbb{Q}$.

   (c) $f(x) = 2f(x + 1) + x + 2$, for every $x \in \mathbb{Q}$.

   Proof: Let $g(x) = f(x)+x$. Then the conditions become $g(x+y) = g(x)g(y)$, $g(1) > 0$, $g(x) = 2g(x + 1)$. From the last one get that $g(1) = 2g(2) = 2g^2(1) \Longrightarrow g(1) = 1/2$. Have $g(1) = g(1)g(0)$ and $g(1) \neq 0$ so $g(0) = 1$. Then get that $g(-x) = 1/g(x)$ from the first relation. So only need to find g on the positive rationals.

   Now $g(n) = g^n(1) = 1/2^n$ for $n \in \mathbb{N}$. $1/2 = g(1) = g(n/n) = g(1/n)^n$ so $g(1/n) = (1/2)^{1/n}$. Then easy to see that $g(x) = g(m/n) = (1/2)^{m/n} = 1/2^x$. And by $g(-x) = 1/g(x)$ get that for all rational x we have $g(x) = 1/2^x$. So $f(x) = -x + \frac{1}{2^x}$. Clearly this satisfies. ■

4. Let $ABCD$ be a rectangle of side lengths m, n made out of $m \times n$ unit squares. Assume that m and n are two odd and coprime positive integers. The points of intersection

between the main diagonal AC and the sides of the unit squares it encounters are $A_1, A_2, \dots, A_k$ in this order ($k \geq 2$), and $A_1 = A$ and $A_k = C$. Prove that

$$A_1A_2 - A_2A_3 + A_3A_4 - \cdots + (-1)^k A_{k-1}A_k = \frac{\sqrt{m^2+n^2}}{mn}.$$

Proof: The general form of a point of intersection between AC and a side of a unit square is $(a, \frac{m}{n}a)$ or $(\frac{n}{m}b, b)$, depending on whether the point is on a vertical or horizontal lattice line. Note that $\gcd(m,n) = 1$ implies that none of these points is lattice.

Two consecutive points on AC (WLOG may say that $AB = m > n = BC$) look either like $(a, \frac{n}{m}a), (a+1, \frac{n}{m}(a+1))$ (in which case the length of the segment is l/m, where $l = \sqrt{m^2+n^2}$), or $(a, \frac{n}{m}a), (\frac{m}{n}b, b)$ (only these two because $m > n$). In this case the segment has length

$$\sqrt{\left(a - \frac{m}{n}b\right)^2 + \left(\frac{n}{m}a - b\right)^2} = \sqrt{a^2l^2/m^2 - 2abl^2/(mn) + b^2l^2/n^2} = l\left|\frac{a}{m} - \frac{b}{n}\right|$$

To a point on a vertical lattice line (called a vertical point) associate the number $a_k = a/m$ and to one on the horizontal lattice line (called a horizontal point) the number $a_k = b/n$. Then the segment $A_kA_{k+1} = l|a_k - a_{k+1}|$. One thing to note here is that $a_k < a_{k+1}$ always.

I still have to write up this solution. I hope to put it up as soon as possible.

■

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